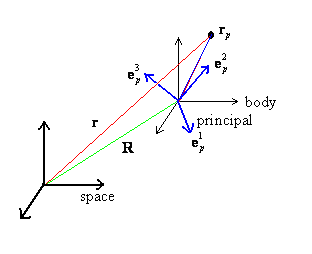
**N2L in Accelerated Reference Frames**

Now we’d like to look another perspective on N2L – that of accelerated reference frames. Sometimes it is more convenient to solve problems by switching reference frames – which don’t necessarily have to be accelerating. Often times problems simplify when expressed from a different reference frame.

**Defining a general accelerated reference frame**

Consider a space reference frame that is ‘stationary’, a body reference frame that translates along a specified trajectory, **R**(t), and principle reference frame whose origin is coincident with the body reference frame, but which rotates at a specified rate, **Ω**, about it. The reason for these names in particular will be elucidated when we describe rigid body rotations – but later on that.



The position of the particle can be specified as:



where **r** is the position of the particle w/r to space, and **r**p is the position w/r to the principal axis. **r**b is the position w/r to the body axes and is the same vector as **r**p since the two axes’ origins are coincident. Of course their components won’t be the same usually.

**N2L in terms of new reference frame coordinates**

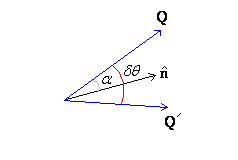
The procedure for translating to a new reference frame is straightforward. Starting from the space axes reference frame, Newton’s second law states,



Now we need to evaluate the derivatives of **r**p(t). So we will write it down in terms of its basis vectors. So (Einstein summation notation with the i’s)…



The first term in the derivative is usually referenced as the velocity of the particle w/r to the rotating reference frame. It is the velocity you would measure if you considered yourself at rest in the rotating frame. Now we need to evaluate the rate of change of the principle axis vectors. Since the principal axes are changing in a known way (they rotate w/r to the body axis at rate **Ω**) so we can certainly ascertain this derivative. To that end, suppose we rotate a vector, **Q**, about axis **n** by an amount δθ.



We can write **Q**′ in terms of **Q** as approximately



This means that we can express the derivative of this vector as:



and so we have:



So therefore we have:



and now we can finish writing our expression for . We have:



Parenthetically, we can generalize this equation to any vector, and write, as is commonly notated:



(the LHS derivative is the rate of change of the vector in the inertial frame, and the RHS derivative is its rate of change as measured in the rotating frame) And now we need one more derivative. So continuing we have (allowing possibility that **Ω** is changing with time so that **α** = d**Ω**/dt):



Now we identify as the giving the components of the acceleration w/r to the principle axis. Similarly, gives the components of the velocity, just as gives the components of the position. And so we can write:



where **a**p, **v**p, are the acceleration and velocity of the particle as measured w/r to the principal axes. And so finally filling this into our N2L equation, and taking everything over to the LHS, except for the m**a**p we get N2L from an accelerated reference frames perspective:

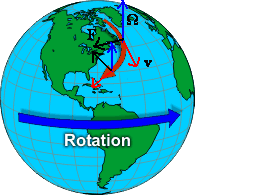


The -m**A** term is called the inertial force that arises from the translationala motion of the reference frame. The other forces arise from the rotation of the reference frame (for instance, if we have a reference frame fixed in orientation w/r to the Earth’s surface, it is simultaneously translating, and rotating – since the reference y-axis, say will be pointing in completely different directions when the Earth has completed half a revolution). The -2m**Ω**×**v** term is the Coriolis force which will appear to push things (at least on Earth) in the direction along the line of the rotation of the Earth. The –m**Ω**×(**Ω**×**r**) term is the centrifugal force, which will appear to push things outward away from the axis of rotation. And the –m**α**×**r** term is the tangential force which appears to push things against the tangential direction familiar from our discussion of circular motion. The utility of such a program is demonstrated if we try to solve for the motion of objects where an accelerated reference frame is a natural frame of reference. Using such notation we could write this equation as:

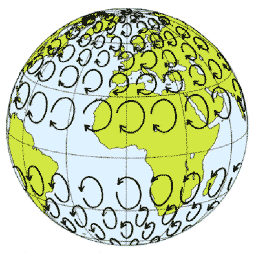


Let’s have a qualitative discussion of each of these forces. First let’s observe that if your mass were zero, then none of these fictitious forces would exist. This is a clue that these aren’t really forces, but rather a result of your own inertia (the fact that the gravitational force F = mg also depends on m is what led Einstein to his general theory of relativity; in essence the gravitational force is also fictitious and is simply a result of the curvature of space-time). So in essence, when you feel these fictitious forces, you’re really just feeling your own inertia.

Let’s discuss these fictitious forces in turn. First, the inertial force **F**inertial = -m**A** is the force that you experience as the origin of your coordinate system is accelerating. For instance, when you sit in your car, and step on the accelerator, you feel a ‘force’ pushing you backward. Again this is just your own inertia, and this is what I’m calling the inertial force. Secondly we have the Coriolis force, **F**coriolis = -2m**Ω**×**v**p. This force is present in a *rotating* reference frame. Notice that when we throw a rock upwards, the Earth will rotate beneath it during its time of flight. Thus the rock will not land at the exact same place it fell. It would appear then, from our perspective, that some force as moved it. This fictitious force, very roughly, is what **F**coriolis represents. The Coriolis force gives rise to some interesting consequences related to meteorology. Consider in the northern hemisphere moving down one of the longitude lines for instance. The Coriolis force on this object, using the right hand rule, would point to the southwest. Therefore the object would tend to turn left. Recalculating the force in this case we would see that the Coriolis force points a little north west. And so the thereupon the object would be forced upward a little. If we continue this analysis, we will see that such an object would be compelled to move in a circle by the Coriolis force. Note too that the Coriolis force gets stronger closer to the poles, in the sense that for motion confined to the tangent plane at the poles, the force is also purely in the tangent plane at that point. But at the equator, again for motion in the tangent plane, the Coriolis force points perpendicular to the tangent plane (i.e. up away from the surface or down into the surface) and so wouldn’t have much affect on the trajectory of objects along the tangent plane. Now the Coriolis force is too weak to significantly affect the measure of macroscopic objects – but it is large enough to affect the motion of the atmosphere.



And one does see its effects during large scale motions of air – like during hurricanes. The Coriolis force rotates these air columns clockwise in the northern hemisphere. Repeating our arguments in the southern hemisphere, you can see that it would rotate them oppositely: counter-clockwise, as illustrated below:

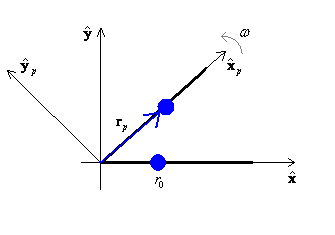


Then, there is the centrifugal force, **F**centrifugal = -m**Ω**×(**Ω**×**r**). If you assume **Ω** is perpendicular to **r**, then this term works out to Fcentrifugal = -mΩ2r = -mv2/r (using Ω = v/r). This is our familiar centripetal force, but with a minus sign, which makes it outward pointing. This is the familiar force you feel when rounding a tight turn.

Last, we have the tangential force **F**tangential = -m**α**×**r**, which only shows up if the angular velocity of the reference frame is changing. This force will appear to acting in the direction opposite to the tangential direction familiar from circular motion.

**Example: Bead on rotating rod**

Consider a bead on a rotating rod, which starts off at the position r0. If the rod rotates counter-clockwise at the rate of ω, what will be the position of the bead as a function of time?



To answer this, we will put ourselves in a reference frame coincident with the space reference frame, but which rotates with rod. Note that the **z** and **z**p axes will always be coincident, and pointing out of the page. In that case N2L will read:



So we have the two equations:



The general solution to the latter is:



Assuming the initial velocity is 0, we’d have:



and so the solution goes to:



Assuming the initial position is r0. Then our solution goes to:

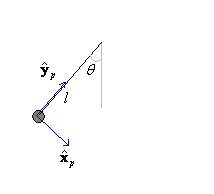


and we’re given the normal force too, which is nice plus.



**Example: N2L from perspective of pendulum**

Consider Tarzan on a swing. What is N2L for Tarzan, from the perspective of his reference frame?



Well we have the following equation. Keep in mind that Tarzan is always at the origin of his reference frame, rather unlike the bead in the previous example. So for Tarzan, his position and velocity are always 0 (w/r to his reference frame of course).



which splits into the equations:



which are indeed the equations as we know.

**Example [actually, this appears to be wrong/inconsistent with next example – coriolis force somehow should be present?]**

Valerie is playing baseball on a space ship in the year 2154. The habital part of the space ship consists of a giant cylinder of radius R and central axis of length L. To provide an artificial sort of gravity, it rotates counter clockwise around its central axis once every hour. Let’s set an x-y coordinate system at the center of the cylinder, fixed in space. And say that Valerie is at the bottom of the cylinder at time t = 0. What are Valerie’s x and y coordinates as a function of time thereafter, relative to the central axis of the spaceship? Suppose at time t = 0, Valerie takes her baseball and tosses it from height h (relative to herself) up into the air with speed (relative to herself) v. What are the x, y coordinates of the baseball as a function of time? What are its coordinates relative to Valerie? Expand the relative coordinates to second order in time (presuming time of flight of ball is much less than space station’s rotational period). What do these equations look like? What is the effective acceleration of ‘gravity’?

Valerie’s position is:



At time t = 0, she tosses a baseball up into the air, from an inital height h, and upward velocity v. Taking account of the implicit horizontal motion of the ball (due to rotation of space station), the position of the ball as a function of time is:



The ball’s coordinates relative to Valerie are:



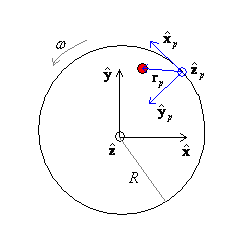
Doing the Taylor series approximation, out to second order:



These are none other than the typical projectile motion equations here on earth, except with an acceleration due to ‘gravity’ of a = -ω2R, instead of g.

**Example: N2L in rotating space station**

Consider a cylindrical space station of radius R, rotating at rate ω. And consider some object in it – say a thrown ball, or whatever. What are the equations of motion of this ball in this reference frame? Note that the z axis and zp axis point out of the page and so they are always the same vector.



We have:



Splitting this up into components, we have:



which breaks up into the equations:



Assume that there is no external force acting on the ball. Then just focusing on the x-y equations we have:



We see that if y is small compared to R, and if dx/dt is negligible, then the y equation of motion looks like free fall with an effective downward acceleration of ω2R. Still, let’s solve these equations if possible. Since they’re a coupled initial value problem, a Laplace transform might be useful. So taking the transform we have:



Grouping the x’s and y’s…



We’ll assume that it starts on the y-axis and is thrown vertically. So we’re assuming x0 = 0, and v0x = 0. In this case we have:



Solving for x(s) and y(s) we get:



and,



And now we’ll have to invert the transform. So we have (from complex analysis)



which gives, upon doing the residues and all…



(note that you can also do this by looking up the required inverse Laplace transforms in a table – or you can do it a completely different way if you desire) Suppose you throw something up in the air at time t = 0 from the origin. How long till it hits the ground? We have, using the y-equation:



Assuming t << 1/ω (meaning that it falls back to the ground long before the space station makes a complete rotation) then we can approximate this as:



Out to first order this comes to:



which is expected from the approximation that ω2R acts approximately as the acceleration due to gravity.

**Perturbative approach to solving equations**

Consider equations again,



We can expand our solution in a Taylor series,



We will recognize a0 and b0 as the initial positions, and a1, b1 as the initial velocities. In this case we can say,



Plugging these expressions into our equations, and keeping terms only up to first order in t, we have:



Grouping powers together,



equating coefficients to zero we have:



So now we know what a2 is, and what b2 is. And knowing those we can get a3 and b3. Plugging back into our expression we have:



which are same as the solutions above, when expanded out to this order in t.